A REMARK ON DEGENERATION OF A COMPACT RIEMANN SURFACE OF GENUS 2

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ABSTRACT

A compact Riemann surface of genus 2, whose period matrix (π_{ij}) is arbitrary, is degenerated, thus removing a restriction on the degeneration in a previous paper by the author.

1.

Recently the author proved, [1], that if a period matrix in genus 2 was degenerated by allowing the period π_{12} to tend to zero, the three "moveable" branch points coalesced yielding a formula which represented the one moveable branch point in genus 1 and, furthermore, the formula was given in terms of π_{11}, the period for the "first" limit Riemann surface. The hypotheses in [1] were that we had both a specified homology basis on S, and that for this homology basis the period π_{12} was small. While the result obtained is correct, it is biased in the sense that there is no *a priori* reason to expect that the period π_{11} should be distinguished.

We wish to remove the restriction that we start with π_{12} small. We will assume that the topological type is preserved till we reach the limit. Thus, we will have at our disposal a topological model with fixed homology basis throughout our discussion. We are thus able to generalize the result in a number of ways. First, we no longer claim that the three "moveable" branch points coalesce but rather just that three of the branch points on S coalesce. It may be the case that two of the moveable branch points tend to 0, 1 or ∞, with the third branch point going to the point $1/\lambda$ in genus 1.

Second, we now have $1/\lambda$ computed as a modular function of π_{11} or of π_{22}, thus the bias is eliminated.

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2.

We start here with the Riemann surface, S, of [1], with homology basis Γ, Δ of fig. 1 in [1]. From S, Γ, Δ, we compute the period matrix (π_{ij}) where now π_{12} is not necessarily near zero.

Now we degenerate the period matrix by letting π_{12} tend to zero. We indicate what happens in two stages.

For some new value of π_{12}, which we will call π'_{12} we know that there is a new homology basis, Γ', Δ' which yields the period matrix (π'_{ij}). For S', Γ', Δ' we obtain new formulas for the branch points as quotients of theta constants.

On S', Γ', Δ' we draw the original homology basis, Γ, Δ, as found in [1]. Now Γ, Δ and Γ', Δ' are related by an element, M', of $Sp(2, z)$ where

$$(\Gamma, \Delta) = M' \cdot (\Gamma', \Delta'),$$

where it is understood that the homology bases are treated as column vectors and $\cdot$ is ordinary matrix multiplication (see [2]).

For S, Γ, Δ, we have computed the values of the branch points as quotients of theta constants in [1], but we now need to know the period matrix (π^*_{ij}), of S', Γ, Δ. We note that the theta characteristics are known in terms of Γ, Δ. Now,

$$(\pi^*_{ij}) = M' \cdot (\pi'_{ij}) = (A'\pi' + B')(C'\pi' + D')^{-1}.$$

Thus we can compute the period matrix (π^*_{ij}) and ask what happens to the branch points in terms of (π^*_{ij}) (not necessarily a diagonal matrix).

Symbolically we can represent the degeneration as follows:

$$\begin{array}{ccc} S, \Gamma, \Delta\,(\pi_{ij}) & \overset{\text{identity}}{\leftrightarrow} & S, \Gamma, \Delta\,(\pi_{ij}) \\ \downarrow & & \downarrow \\ S', \Gamma, \Delta\,(\pi^*_{ij}) & \overset{M'}{\leftrightarrow} & S', \Gamma', \Delta'(\pi'_{ij}) \\ \downarrow & & \downarrow \\ S'', \Gamma, \Delta\,(\pi^{**}_{ij}) & \overset{M''}{\leftrightarrow} & S'', \Gamma'', \Delta''(\pi''_{ij}), \end{array}$$

where π''_{12} is now near zero and the middle line represents an intermediate stage in the degeneration.

Thus to discover the effect, on the Riemann surface, of π_{12} going to zero, we study the effect of the period matrix, (π^{**}_{ij}), as a function of π_{12}, on the formulas of th. 1 of [1].

A set of generators for $Sp(2, z)$ is given in [2], th. C. One computes the effect of these generators on (π''_{ij}) and we obtain:

THEOREM. *Let S, Γ, Δ with the period matrix (π_{ij}) degenerate by allowing π_{12} to go to zero. Then three of the branch points on S coalesce, yielding a limit surface of genus 1 with branch points over $0, 1, \infty$ and with fourth branch point, $1/\lambda$, given by*

$$1/\lambda = \frac{\theta^4\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)}{\theta^4\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)},$$

where τ is a modular function of π_{11} or of π_{22}.

We have a function of π_{11} or of π_{22} since under the action of M'' the period π_{11} is no longer distinguished. Put another way, π_{11} was distinguished in [1] by the choice of homology basis and the fact that we assumed π_{12} was already small. Now, with π''_{12} small we are dealing with that homology basis, but for a different period matrix.

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